HW 1
Topics in Computation Theory: Geometric Approximation Algorithms (CS700)

Due date: May 29

May 15, 2007

1 Min disk

Exercise 1.1 (Compute clustering radius.) [10 Points]
Let C and P be two given sets of points in the plane, such that $k = |C|$ and $n = |P|$. Let $r = \max_{p \in P} \min_{c \in C} \|c - p\|$ be the ***covering radius*** of P by C (i.e., if we place a disk of radius r around each point of C all those disks covers the points of P).

(A) Give a $O(n + k \log n)$ expected time algorithm that outputs a number α, such that $\alpha \leq r \leq 10\alpha$.

(B) For $\varepsilon > 0$ a prescribed parameter, give a $O(n + k\varepsilon^{-2} \log n)$ expected time algorithm that outputs a number α, such that $\alpha \leq r \leq (1 + \varepsilon)\alpha$.

Exercise 1.2 (Randomized k-enclosing disk.) [5 Points]
Given a set P of n points in the plane, and parameter k, present a (simple) randomized algorithm that computes, in expected $O(n(n/k))$ time, a circle D that contains k points of P, and radius$(D) \leq 2r_{\mathrm{opt}}(\mathsf{P}, k)$.

(This is a faster and simpler algorithm than the one presented in the class notes.)

2 Quadtree

Exercise 2.1 (Quadtree for fat quadtrees.) [5 Points]
A triangle $\triangle$ is called α-fat if each one of its angles is at least α, where $\alpha > 0$ is a prespecified constant (for example, α is 5 degrees). Let P be a triangular planar map of the unit square (i.e., each face is a triangle), where all the triangles are fat, and the total number of triangles is n. Prove that the complexity of the quadtree constructed for P is $O(n)$.

Exercise 2.2 (Quadtree construction is tight.) [5 Points]
Prove that the bounds on the size of a quadtree are tight. Namely, show that for any $r > 2$ and any positive integer $n > 2$, there exists a set of n points with diameter $\Omega(1)$ and spread $\Phi(P) = \Theta(r)$, and such that its quadtree has size $\Omega(n \log \Phi(P))$.

Exercise 2.3 (Space filling curve.) [10 Points]

The ***Peano curve*** $\sigma : [0, 1) \to [0, 1)^2$, maps a number $\alpha = 0.t_1t_2t_3\ldots$ (the expansion is in base 3) to the point $\sigma(\alpha) = (0.x_1x_2x_3\ldots,\ 0.y_1y_2\ldots)$, where $x_1 = t_1$, $x_i = \phi(t_{2i-1}, t_2+t_4+\ldots+t_{2i-2})$, for $i \geq 1$. Here, $\phi(a, b) = a$ if b is even and $\phi(a, b) = 2 - a$ if b is odd. Similarly, $y_i = \phi(t_{2i}, t_1 + t_3 + \cdots + t_{2i-1})$, for $i \geq 1$.

(A) Prove that the mapping σ covers all the points in the open square $[0, 1)^2$, and it is one to one.

(B) Prove that σ is continuous.

3 Will-separated pairs decomposition

Exercise 3.1 (WSPD Structure.) [5 Points]

(A) Show an example of a point set and its associated WSPD, so that a single set participates in $\Omega(n)$ sets.

(B) Show, that if we list explicitly the sets forming the WSPD (even if we show each set exactly once) then the total size of such a description is quadratic. (Namely, the implicit representation we use is crucial to achieve efficient representation.)

Exercise 3.2 (Number of resolutions that matter.) [4 Points]

Let P be a n-point set in $\mathbb{R}^d$, and consider the set $U = \left\{ i \,\middle|\, 2^i \leq \|\mathsf{p} - \mathsf{q}\| \leq 2^{i+1},\ \text{for } \mathsf{p}, \mathsf{q} \in \mathsf{P} \right\}$. Prove that $|U| = O(n)$ (the constant depends on d). Namely, there are only n different resolutions that "matter".

Exercise 3.3 (WSPD and sum of distances.) [5 Points]

Let P be a set of n points in $\mathbb{R}^d$. The ***sponginess***[1] of P is the quantity $X = \sum_{\{\mathsf{p},\mathsf{q}\}\subseteq \mathsf{P}} \|\mathsf{p} - \mathsf{q}\|$. Provide an efficient algorithm for approximating X. Namely, given P and a parameter $\varepsilon > 0$ it outputs a number Y such that $X \leq Y \leq (1 + \varepsilon)X$.

(The interested reader can also verify that computing (exactly) the sum of all ***squared*** distances (i.e., $\sum_{\{\mathsf{p},\mathsf{q}\}\subseteq \mathsf{P}} \|\mathsf{p} - \mathsf{q}\|^2$) is considerably easier.)

4 Brunn-Minkowski inequality

Exercise 4.1 (Boxes can be separated.) [1 Points]

(Easy.) Let A and B be two axis-parallel boxes that are interior disjoint. Prove that there is always an axis-parallel hyperplane that separates the interior of the two boxes.

Exercise 4.2 (Brunn-Minkowski inequality slight extension.) [4 Points]

> **Corollary 4.3** *For A and B compact sets in $\mathbb{R}^n$, we have for any $\lambda \in [0, 1]$ that* $\mathrm{Vol}(\lambda A + (1 - \lambda)B) \geq \mathrm{Vol}(A)^{\lambda}\,\mathrm{Vol}(B)^{1-\lambda}$.

[1] Also known as the sum of pairwise distances in the literature, for reasons that I can not fathom.