Spring Semester 2005

Topics in Computation Theory (CS700)

Discrete Geometry

Homework 2

This homework is a bit longer, and you have more time for it. There is no class on March 15 and 17, so instead work on these problems. This homework is due on **March 22**, at the beginning of the class.

On the top of the first sheet that you turn in, please put (a) your name and student number, (b) how much time you spent working on the homework, and (c) a little table with your self-evaluation as explained on the course webpage.

1. Prove Carathéodory's theorem: Let $X \subset \mathbb{R}^d$, and let $x \in \text{conv}(X)$. Then we can express x as a convex combination of at most $d+1$ elements of X:

$$x = \sum_{i=1}^{d+1} t_i x_i,$$

where $t_i \geq 0$, $\sum_{i=1}^{d_1} t_i = 1$, and $x_i \in X$. (Hint: Use Radon's lemma).

2. Find an example of four convex sets in the plane such that the intersection of any three of them contains a segment of length one, but the intersection of all four does not contain a segment of length one.

3. A *strip of width* w is a part of the plane bounded by two parallel lines at distance w. The *width* of a set $X \subseteq \mathbb{R}^2$ is the smallest width of a strip containing X.

   (a) Prove that a compact convex set of width one contains a segment of length one of every direction.

   (b) Let $\{C_1, C_2, \ldots, C_n\}$ be closed convex sets in the plane, $n \geq 3$, such that the intersection of every three of them has width at least one. Prove that $\bigcap_{i=1}^n C_i$ has width at least one.

4. Let A be a set of $d+2$ points in $\mathbb{R}^d$. We call $x \in \mathbb{R}^d$ a *Radon point* of A if x is contained in the convex hull of two disjoint subsets of A. Prove that if no $d+1$ points of A are affinely dependent (that is, lie in the same hyperplane), then the Radon point of A is unique.

5. A soccer ball consists of black hexagons (6-gons) and white pentagons (5-gons). Each hexagon is adjacent to three pentagons and three hexagons, each pentagon is adjacent to five hexagons. How many pentagons and hexagons are there?

6. Let $n_1, n_2, \ldots, n_k$ be natural numbers (without zero), and let $n_1 + n_2 + \cdots + n_k = n$. We want to prove that $\sum_{i=1}^{k} n_i^2 \le (n-k+1)^2 + k - 1$.

   (a) Explain why the following "solution" is not a correct proof: In order to make $\sum_{i=1}^{k} n_i^2$ as large as possible, we must set all the n_i but one to 1. The remaining one is therefore $n-k+1$, and in this case the sum of squares is $(n-k+1)^2+k-1$.

   (b) Give a correct proof.

7. Let $A_1, A_2, \ldots, A_k \subset \mathbb{R}^d$ be finite sets, where $1 \le k \le d$. Prove that there exists a $(k-1)$-flat (that is, an affine subspace of dimension $k-1$), such that every hyperplane containing it has at least $\frac{1}{d+1}|A_i|$ points of A_i in both of its closed half-spaces, for all $i = 1, 2, \ldots, k$.