

Weighted vertex cover

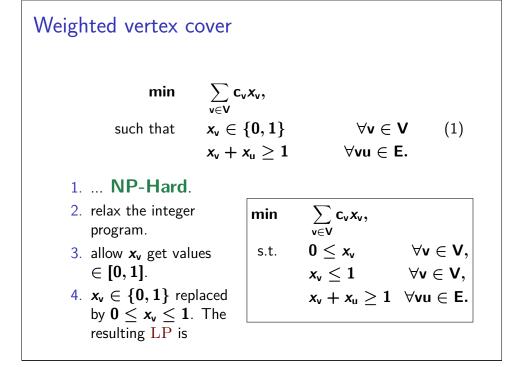
Weighted Vertex Cover problem

 $\label{eq:G} \begin{array}{l} \mathsf{G} = (\mathsf{V},\mathsf{E}). \\ \text{Each vertex } \mathsf{v} \in \mathsf{V}: \mbox{ cost } \mathsf{c}_{\mathsf{v}}. \\ \text{Compute a vertex cover of minimum cost.} \end{array}$

1. vertex cover: subset of vertices \mathbf{V} so each edge is covered.

2. NP-Hard

- 3. ...unweighted Vertex Cover problem.
- 4. ... write as an integer program (IP):
- 5. $\forall \mathbf{v} \in \mathbf{V}: x_{\mathbf{v}} = 1 \iff \mathbf{v}$ in the vertex cover.
- 6. $\forall \mathsf{vu} \in \mathsf{E}$: covered. $\implies x_{\mathsf{v}} \lor x_{\mathsf{u}}$ true. $\implies x_{\mathsf{v}} + x_{\mathsf{u}} \ge 1$.
- 7. minimize total cost: $\min \sum_{v \in V} x_v c_v$.



Weighted vertex cover - rounding the LP

- 1. Optimal solution to this LP: $\widehat{x_{v}}$ value of var X_{v} , $\forall v \in V$.
- 2. optimal value of LP solution is $\hat{\alpha} = \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{c}_{\mathbf{v}} \widehat{\mathbf{x}_{\mathbf{v}}}$.
- 3. optimal integer solution: $\mathbf{x}'_{\mathbf{v}}, \forall \mathbf{v} \in \mathbf{V}$ and α' .
- 4. Any valid solution to IP is valid solution for LP!
- 5. $\hat{\alpha} \leq \alpha'$. Integral solution not better than LP.
- 6. Got fractional solution (i.e., values of $\widehat{x_v}$).
- 7. Fractional solution is better than the optimal cost.
- 8. Q: How to turn fractional solution into a (valid!) integer solution?
- 9. Called *rounding*.

How to round?

- 1. consider vertex **v** and fractional value $\widehat{x_{v}}$.
- 2. If $\widehat{x_v} = 1$ then include in solution!
- 3. If $\widehat{x_v} = 0$ then do not include in solution.
- 4. if $\widehat{x_v} = 0.9 \implies \text{LP}$ considers **v** as being **0.9** useful.
- 5. The $\ensuremath{\mathrm{LP}}$ puts its money where its belief is...
- 6. ... $\hat{\alpha}$ value is a function of this "belief" generated by the LP.
- 7. Big idea: Trust LP values as guidance to usefulness of vertices.
- 8. Pick all vertices \geq threshold of usefulness according to LP.
- 9. $S = \{ \mathbf{v} \mid \widehat{x_{\mathbf{v}}} \geq 1/2 \}.$
- 10. Claim: \mathbf{S} a valid vertex cover, and cost is low.
- 11. Indeed, edge cover as: $\forall \mathbf{vu} \in \mathbf{E}$ have $\widehat{x_{\mathbf{v}}} + \widehat{x_{\mathbf{u}}} \geq 1$.
- 12. $\widehat{x_v}, \widehat{x_u} \in (0, 1)$
 - $\implies \widehat{x_{\mathsf{v}}} \geq 1/2 \; {\mathsf{or}} \; \widehat{x_{\mathsf{u}}} \geq 1/2.$
 - \implies **v** \in **S** or **u** \in **S** (or both).
 - \implies **S** covers all the edges of **G**.

The lessons we can take away

Or not - boring, boring, boring.

- 1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
- 2. Not aware of any other **2**-approximation algorithm does not use LP. (For the weighted case!)
- 3. Solving a *relaxation* of an optimization problem into a LP provides us with insight.
- 4. But... have to be creative in the rounding.

Cost of solution

Cost of \boldsymbol{S} :

$$\mathbf{c}_{\mathcal{S}} = \sum_{\mathsf{v} \in \mathcal{S}} \mathbf{c}_{\mathsf{v}} = \sum_{\mathsf{v} \in \mathcal{S}} \mathbf{1} \cdot \mathbf{c}_{\mathsf{v}} \le \sum_{\mathsf{v} \in \mathcal{S}} 2\widehat{x_{\mathsf{v}}} \cdot \mathbf{c}_{\mathsf{v}} \le 2\sum_{\mathsf{v} \in \mathsf{V}} \widehat{x_{\mathsf{v}}} \mathbf{c}_{\mathsf{v}} = 2\widehat{\alpha} \le 2\alpha'$$

since $\widehat{x_v} \ge 1/2$ as $v \in S$. α' is cost of the optimal solution \implies

Theorem

The **Weighted Vertex Cover** problem can be **2**-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.

