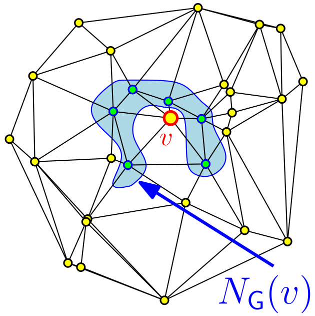
What if the vertex cover is small?

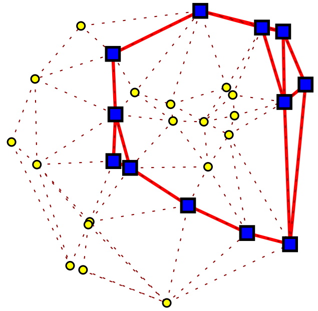
1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with n vertices
2. $K \leftarrow$ Approximate VertexCoverMin up to a factor of two.
3. Any vertex cover of G is of size $\geq K/2$.
4. Naively compute optimal in $O\left(n^{K+2}\right)$ time.

Induced subgraph

Definition
$N_G(v)$: ***Neighborhood*** of v – set of vertices of $\mathbf{G}$ adjacent to v.



Definition
Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph. For a subset $S \subseteq \mathbf{V}$, let $\mathbf{G}_S$ be the ***induced subgraph*** over S.



Exact fixed parameter tractable algorithm

Fixed parameter tractable algorithm for **VertexCoverMin**.

Computes minimum vertex cover for the induced graph $\mathbf{G}_X$:

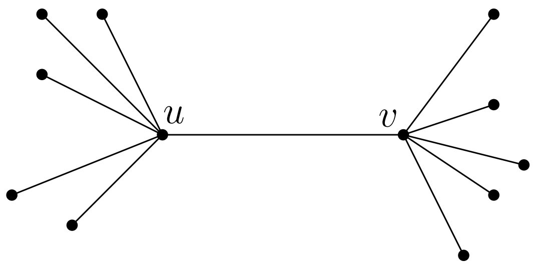
fpVCI (X, β)
    // β: size of VC computed so far.
    if $X = \emptyset$ or $\mathbf{G}_X$ has no edges **then return** β
    e $\leftarrow$ any edge uv of $\mathbf{G}_X$.
    $\beta_1 =$ **fpVCI** $\left(X \setminus \{u, v\}, \beta + 2\right)$
    $\beta_2 =$ **fpVCI** $\left(X \setminus \left(\{u\} \cup N_{\mathbf{G}_X}(v)\right), \beta + |N_{\mathbf{G}_X}(v)|\right)$
    $\beta_3 =$ **fpVCI** $\left(X \setminus \left(\{v\} \cup N_{\mathbf{G}_X}(u)\right), \beta + |N_{\mathbf{G}_X}(u)|\right)$
    return $\min(\beta_1, \beta_2, \beta_3)$.

algFPVertexCover $(\mathbf{G} = (\mathbf{V}, \mathbf{E}))$
    return fpVCI$(\mathbf{V}, 0)$

Depth of recursion

Lemma
The algorithm **algFPVertexCover** *returns the optimal solution to the given instance of VertexCoverMin.*

Proof...



Depth of recursion II

Lemma

The depth of the recursion of **algFPVertexCover**$(\mathbf{G})$ *is at most* α, *where* α *is the minimum size vertex cover in* $\mathbf{G}$.

Proof.

1. When the algorithm takes both u and v - one of them in opt. Can happen at most α times.
2. Algorithm picks $N_{G_X}(v)$ (i.e., β_2). Conceptually add v to the vertex cover being computed.
3. Do the same thing for the case of β_3.
4. Every such call add one element of the opt to conceptual set cover. Depth of recursion is $\leq \alpha$.

□

Vertex Cover

Exact fixed parameter tractable algorithm

Theorem

$\mathbf{G}$: *graph with* n *vertices. Min vertex cover of size* α. *Then,* **algFPVertexCover** *returns opt. vertex cover.*
Running time is $O\left(3^{\alpha} n^2\right)$.

Proof:

1. By lemma, recursion tree has depth α.
2. Rec-tree contains $\leq 2 \cdot 3^{\alpha}$ nodes.
3. Each node requires $O\left(n^2\right)$ work. ■

Algorithms with running time $O(n^c f(\alpha))$, where α is some parameter that depends on the problem are ***fixed parameter tractable***.