The Josephus Game

Given n player sitting in a circle, and a number m.

A hot potato starts at player 1, and is passed around m times. The player holding the potato then is eliminated, the next player gets the potato, and the game continues until only one player is left.

$n = 6,\ m = 2$

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Can we do it in less than quadratic time?

Our linked-list Josephus program needs $(n-1)m$ link transversals.
Can we do it more efficiently?

First observation: If we are currently at position p, then after m passes we will be at position $(p+m) \bmod n$ (positions numbered from 0 to $n-1$).

Difficulty: How can we maintain the names of the people remaining in the game?

We need a data structure that stores a sequence of n elements, and supports one main operation: Remove the kth element.

No standard Python data structure supports this operation efficiently. We need to implement it ourselves...

The Rank Tree

Supports the following operations:

- Construct from an array with n elements;
- `find(k)` returns the item at rank (index) k;
- `remove(k)` removes the element at rank k;
- `size()` returns the current size.

Idea: Store the elements in a binary tree in in-order sequence. Store in each node t the size of the subtree whose root is t.

To find the node with rank k, we just have to follow a path from the root.

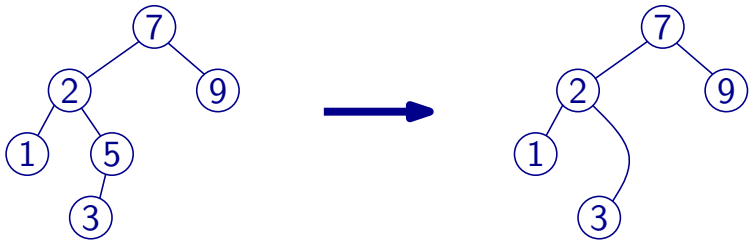
The remove operation

How to remove the node t with rank k?

First find the node t of rank k.
Then there are three cases:

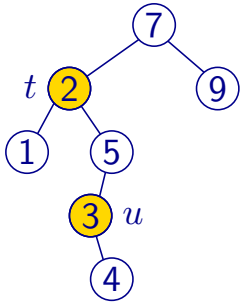
1. Easy case: t is a leaf node.

2. Slightly harder case: t has one child



The remove operation

3. If t has two children, then find the leftmost node u in the right subtree of t. Replace the element stored at t with the element from u. Finally, remove the node u.



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The remove operation

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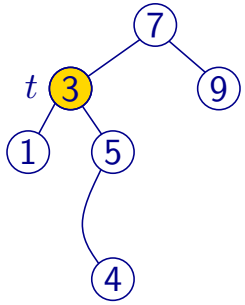

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Rank tree analysis

`find` and `remove` take time $O(h)$, where h is the height of the tree.

When we construct the tree, we can make a perfectly balanced tree.
Its height is $\lceil \log(n+1) \rceil - 1$.

Therefore `find` and `remove` take time $O(\log n)$, and the total running time for the Josephus problem is $O(n \log n)$.